Fractals, Turtles and Snowflakes

We want to challenge the myth that this "stuff" is for young children. In this series of articles by Richard Noss we revive some of the ideas that made Logo such an exciting phenomenon when it first appeared. Does it still have anything to offer? Judge for yourself.

Imagine a very long thread. Pull it out into a straight line. Now imagine that a small creature (lets call it a turtle), lives on the thread. It is constrained to move only forwards or backwards – albeit quite a long way forwards or backwards. The thread is a one-dimensional world for the creature.

Now crinkle the thread up. The turtle is still constrained to walk along a one dimensional path, even though that path now exists in (at least) two dimensions. But the turtle does not know that. It can still only walk forwards and backwards. Now crinkle the thread even more (remember that it is very long) so that it bends so much that it fills ip the two dimensional space it lives in. Can this be done? And if so, is the turtle's world now one dimensional, or two dimensional or somewhere in between?

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A curve made by crinkling to extreme lengths is extremely badly behaved. That is, the normal theorems of geometry could be expected not to hold for such cases (is it smooth enough anywhere for its slope to be defined?). Curves with similar behaviour have been recognized and studied since the start of the twentieth century by mathematicians such as Hilbert, Hausendorff, and Cantor. As always in such circumstances those cases were outlawed as special 'pathological' cases, and redefined out of the existing theorems of the time. The legacy to later generations of mathematicians was to name the special cases after their inventor, or to employ the inventors name to exclude such cases (or both).

It often occurs in mathematics that the pathological case of yesterday becomes the cornerstone of today. One well-known example is that of non-Euclidian geometry which has developed form the status of an interesting diversion to the basis of differential geometry and relativity. A much more recent development, of which the turtle's thread provides an example, is that of fractal geometry.

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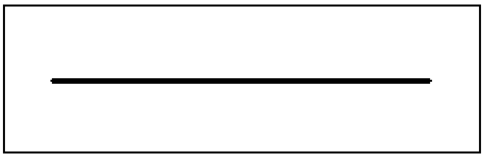
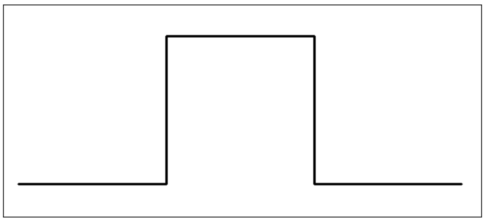
To understand the basic idea of fractal geometry, we need to return to the turtle. Let us suppose that the turtle wants to measure the length of the line of thread. Obviously, what it needs is a measuring instrument, which – as the line is a one-dimensional object – needs to be a unit of length. It would be best if the unit was not too long otherwise the turtle would have trouble with the bends. Let's call the length of the measuring unit ***r*** (for ruler) turtle units. If the section of line to be measured was 1 light-turtle (a large unit of turtle length), there would be 1/r equal pieces each of length ***r***

What if the curve bends too much? The obvious solution would be to decrease the unit length of measure (*r*). If the thread is genuinely one-dimensional, haling the unit of measure would simply double the number of equal pieces… and so on. But now comes the problem. Suppose the thread becomes unsmooth, that is impossible ever find a measuring unit small enough to measure with – i.e. to treat the thread as a straight line: Worse, suppose the curve bends so much that the number of pieces (i.e. the length) increases by a factor which is more than the factor by which the unit decreases? Can this happen?

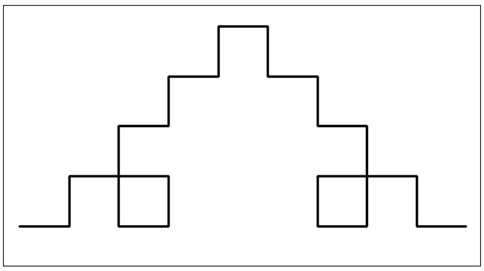
Let us take an example. Suppose we start with a straight line of length one light-turtle.



In this case we could use a measuring unit of length 1 (***r = 1 light turtle***). Now introduce a wrinkle (a rather regular wrinkle) in the line:



In this case the unit of measure needs to be decreased by a factor of 3 (1/3 light-turtles), while the number of segments has increased from 1 to 5. Now introduce another wrinkle.



Now the unit of measure has decreased to 1/9 light turtles (reduced by a factor of 3), while the number of pieces has increased to 25 a factor of 5.

Obviously there is still no difficulty… provided that the process comes to an end. All that needs to be done is to choose smaller and smaller units of measure. But suppose the process just continues. In this case it is clear that the length is increasing at a more rapid rate than the decrease in the measuring instrument – a very strange situation.

Before suggesting a resolution to the problem, let me borrow an idea from the famous mathematics educator George Polya. When in difficulty solving a problem, he counsels the following advice: Do we know a similar problem? Is there an instance that we have come across where an increase in the dimensions (that is going to turn out to be an unfortunate choice of word) of an object brings about a larger change in its measure? In one dimension, this seems strange – as the wrinkled curve above shows. But what about in two dimensions? Here there is an obvious analogy. Take a square, double its sides and its are is not doubled but quadrupled.

How does this help with the turtle curve? The answer lies in seeing what it is that makes the square's area increase faster than its length – namely that its dimension is 2 (rather than 1). I is a property of two-dimensional shapes that their area (measure) increases as the square of the length. For a three dimensional shape, it increases as the cube (and so on (?)).

Now back to the crinkled thread curve. As the length of its unit of measure decreases by a factor of 3, the number of such units increases by a factor of 5. For 'ordinary' straight lines, the number of equal pieces (***N***) of length **r** is ***1/r***. For two dimensional objects, ***N= (1/r)***^{2}. For our crinkled thread curve, ***N = (1/R)***$^{5/3}$. It looks very much like 5/3 is playing the same role as the dimension does in more conventional cases. This seems intuitively sensible, as the more crinkled curve becomes, the more it behaves like a two-dimensional object – in this sense of covering two-dimensional space. If the power to which ***(1/r)*** is raised is called ***D***, the equation for ***D***, the fractal dimension becomes:

$$N = \left(\frac{1}{R}\right)^{D}$$

$$D = \frac{Log\ N}{Log\ {}^{1}\!/_{r}}$$

For the crinkly curve, the fractal dimension would be ***log 5 / log 3***, or about 1.5. Intuitively, a curve which twisted as much as this one, would have a dimension about 'half way' between one and two dimensions.

So what does the curve look like? By far the easiest way to describe the curve is to write a computer program for it. I have chosen to write it in Logo, which is the language I know best, but any language whish allows full recursion (the ability to call a procedure as a sub procedure of itself) would do.

To draw the basic curve is straightforward: I've called it Koch, in honour of the mathematician who invented it:

```
TO    KOCH        : LENGTH
      FORWARD     : LENGTH
      LEFT        90
      FORWARD     : LENGTH
      RIGHT       90
      FORWARD     : LENGTH
      RIGHT       90
      FORWARD     : LENGTH
      LEFT 90
      FORWARD     : LENGTH
END
```

Now consider the structure of the curve. The point is that instead of just going forward, we have to do a KOCH. Just replacing FORWARD by KOCH would almost do it. But of course, the whole point is that the amount to go forward decrease a factor of 3 each time so we have:

```
TO    KOCH        : LENGTH
      FORWARD     : LENGTH/3
      LEFT        90
      FORWARD     : LENGTH/3
      RIGHT       90
      FORWARD     : LENGTH/3
      RIGHT       90
      FORWARD     : LENGTH/3
      LEFT 90
      FORWARD     : LENGTH/3
END
```

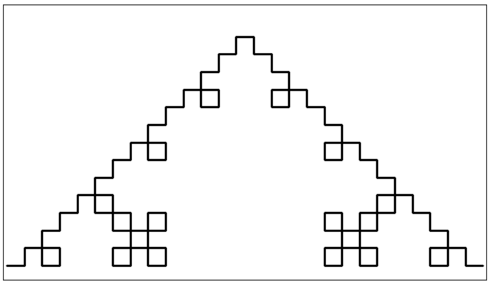
There is only one outstanding problem: that is that nothing actually happens. At the second line of the program the length will continually be divided by 3 (which is the limit, of course, is what happens). This goes on forever, which makes it extremely small, but does not actually drawing anything. What is needed is some kind of limiting factor, beyond which the program will call a halt, draw a line of the required length and execute the remainder of the program. This can be done in many ways. One is simply to specify a minimum length. For example, if a new second line was inserted as follows:

```
IF   :LENGTH < 3     [FORWARD :LENGTH STOP]
```

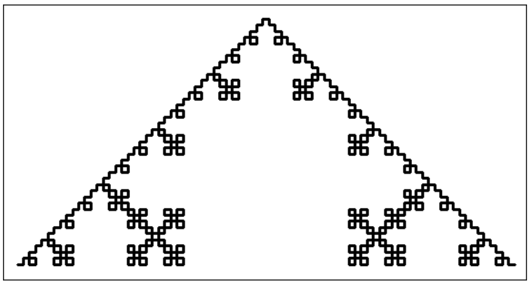
Then KOCH 81 would draw four 'levels' of the curve (81, 27, 9, 3). Another way would be to introduce the limiting value explicitly, which would produce the following completed program:

```
TO    KOCH        : LENGTH : LIMIT
      IF          : LENGTH < : LIMIT [FORWARD : LENGTH STOP]
      FORWARD     : LENGTH/3 : LIMIT
      LEFT        90
      FORWARD     : LENGTH/3 : LIMIT
      RIGHT       90
      FORWARD     : LENGTH/3 : LIMIT
      RIGHT       90
      FORWARD     : LENGTH/3 : LIMIT
      LEFT 90
      FORWARD     : LENGTH/3 : LIMIT
END
```

Trying a few Koch curves rapidly gives credence to the idea of a dimension between 1 and 2. Some examples are provided in the accompanying figures.



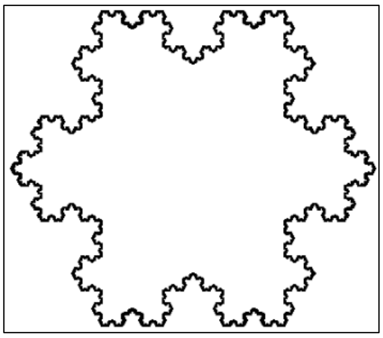
KOCH 81 9



KOCH 81 3

Modifying the KOCH curve can produce some interesting results. For example, it can be modified so that it becomes genuinely space-filling; in this case its fractal dimension is 2. It would spoil the fun to provide the program.

One of the most sticking features of this kind of fractal is that it is the same at each level of magnification. A strikingly beautiful example of such a curve is the 'snowflake'. The picture opposite is a snowflake curve with four levels; again the program is left as an exercise for the reader (actually, it is one program repeated 3 times).



Randomness

In order to appreciate the power of the idea of fractals, it is necessary to introduce one more ingredient – that of randomness. When a random element is introduced, it turns out that many, if not most, natural phenomena can be modeled by exactly the kind of process we have just seen. What, for example, is the length of the edge of the page you are reading [assuming you have printed it out]. Before you get out your ruler thing of the turtle on its thread.

Random fractals are not precisely self-similar like the turtle-based examples above. But instead, they provide models of process as diverse as mountain ranges, the bark of trees, the path of a bolt of lightening and the shape of coastlines (a well-known starting point for considering fractals is to ask just how long is the coastline of Britain?).

Since the introduction of powerful computers, it has been possible to model many complex processes using the idea of fractals. Looking at the pictures on the page and the cover may inspire delight, even awe. But the point is that these are not pictures built by the creativity of human beings (in the sense of an artist, or a computer programmer, 'putting' points of light one by one on a computer screen), they are built from precise mathematical models. That is, despite all intuition, the pictures are produced according to the rules of fractal geometry. Of course the parameters of the models, and the models themselves were set up by programmers, but the pictures were *generated* by the model. Obviously, these models are vastly complex in comparison with the simple examples of fractals we have seen above. The most obvious generalisation is to move beyond the restriction of dimensions between 1 and 2. But they are all between 1 and 2. But they are all based on the same idea, the idea of fractal geometry.

"…I claim that many patterns of Nature are so irregular and fragmented, that compared with Euclid… Nature exhibits not simple a higher degree but an altogether different level of complexity. The number of distinct scales of length of natural patterns is for all practical purposes infinite.

Fractal geometry reveals that some of the most austerely formal chapters of mathematics had a hidden face: a world of pure paltic beauty unsuspected until now".

Benoit Mandlebrot 1977

The ideas in this article are based on the following sources:

1. Mandlebrot B, (1982) The Fractal Geometry of Nature. W.H. Freeman & Co.

2. Abelson, H. and DiSessa, A. (1980) Turtle Geometry, MIT Press.

3. Thornburg, D. (1983) Discovering Apple Logo: an invitation to the art and pattern of nature. Addison Wesley

Although Roamer currently does support variables, students can explore some of the ideas inherent in fractals. See **Fractals** in the Roamer Activity Library.